# Some properties of the inverse error function

Diego Dominici

Abstract. The inverse of the error function, $\operatorname{inverf}(x)$, has applications in diffusion problems, chemical potentials, ultrasound imaging, etc. We analyze the derivatives $\frac{d^n}{dz^n} \operatorname{inverf}(z)\Big|_{z=0}$, as $n \to \infty$ using nested derivatives and a discrete ray method. We obtain a very good approximation of $\operatorname{inverf}(x)$ through a high-order Taylor expansion around $x = 0$. We give numerical results showing the accuracy of our formulas.

## 1. Introduction

The error function $\operatorname{erf}(z)$, defined by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp\left(-t^2\right) dt,$$

occurs widely in almost every branch of applied mathematics and mathematical physics, e.g., probability and statistics [**Wal50**], data analysis [**Her88**], heat conduction [**Jae46**], etc. It plays a fundamental role in asymptotic expansions [**Olv97**] and exponential asymptotics [**Ber89**].

Its inverse, which we will denote by $\operatorname{inverf}(z)$,

$$\operatorname{inverf}(z) = \operatorname{erf}^{-1}(z),$$

appears in multiple areas of mathematics and the natural sciences. A few examples include concentration-dependent diffusion problems [**Phi55**], [**Sha73**], solutions to Einstein's scalar-field equations [**LW95**], chemical potentials [**TM96**], the distribution of lifetimes in coherent-noise models [**WM99**], diffusion rates in tree-ring chemistry [**BKSH99**] and $3D$ freehand ultrasound imaging [**SJEMFAL$^+$03**].

Although some authors have studied the function $\operatorname{inverf}(z)$ (see [**Dom03b**] and references therein), little is known about its analytic properties, the major work having been done in developing algorithms for numerical calculations [**Fet74**]. Dan Lozier, remarked the need for new techniques in the computation of $\operatorname{inverf}(z)$ [**Loz96**].

In this paper, we analyze the asymptotic behavior of the derivatives $\frac{d^n}{dz^n} \operatorname{inverf}(z)\Big|_{z=0}$ for large values of $n$, using a discrete WKB method [**CC96**]. In Section 2 we present

1991 *Mathematics Subject Classification.* Primary 33B20; Secondary 30B10, 34K25.

*Key words and phrases.* Inverse error function, asymptotic analysis, discrete ray method, differential-difference equations, Taylor series.

This work was partially supported by a Provost Research Award from SUNY New Paltz.

some properties of the derivatives of $\operatorname{inverf}(z)$ and review our previous work on nested derivatives. In Section 3 we study a family of polynomials $P_n(x)$ associated with the derivatives of $\operatorname{inverf}(z)$, which were introduced by L. Carlitz in [**Car63**]. Theorem 3.3 contains our main result on the asymptotic analysis of $P_n(x)$. In Section 4 we give asymptotic approximations for $\frac{d^n}{dz^n} \operatorname{inverf}(z)\big|_{z=0}$ and some numerical results testing the accuracy of our formulas.

## 2. Derivatives

Let us denote the function $\operatorname{inverf}(z)$ by $\mathfrak{I}(z)$ and its derivatives by

$$d_n = \frac{d^n}{dz^n} \operatorname{inverf}(z)\bigg|_{z=0}, \quad n = 0, 1, \ldots. \tag{2.1}$$

Since $\operatorname{erf}(z)$ tends to $\pm 1$ as $z \to \pm\infty$, it is clear that $\operatorname{inverf}(z)$ is defined in the interval $(-1, 1)$ and has singularities at the end points.

Proposition 2.1. *The function $\mathfrak{I}(z)$ satisfies the nonlinear differential equation*

$$\mathfrak{I}'' - 2\mathfrak{I}\left(\mathfrak{I}'\right)^2 = 0 \tag{2.2}$$

*with initial conditions*

$$\mathfrak{I}(0) = 0, \quad \mathfrak{I}'(0) = \frac{\sqrt{\pi}}{2}. \tag{2.3}$$

Proof. It is clear that $\mathfrak{I}(0) = 0$, since $\operatorname{erf}(0) = 0$. Using the chain rule, we have

$$\mathfrak{I}'\left[\operatorname{erf}(z)\right] = \frac{1}{\operatorname{erf}'(z)} = \frac{\sqrt{\pi}}{2} \exp\left\{\mathfrak{I}^2\left[\operatorname{erf}(z)\right]\right\}$$

and therefore

$$\mathfrak{I}' = \frac{\sqrt{\pi}}{2} \exp\left(\mathfrak{I}^2\right). \tag{2.4}$$

Setting $z = 0$ we get $\mathfrak{I}'(0) = \frac{\sqrt{\pi}}{2}$ and taking the logarithmic derivative of (2.4) the result follows. □

To compute higher derivatives of $\mathfrak{I}(z)$, we begin by establishing the following corollary.

Corollary 2.2. *The function $\mathfrak{I}(z)$ satisfies the nonlinear differential-integral equation*

$$\mathfrak{I}'(z) \int_0^z \mathfrak{I}(t)dt = -\frac{1}{2} + \frac{1}{\sqrt{\pi}}\mathfrak{I}'(z). \tag{2.5}$$

Proof. Rewriting (2.2) as

$$\mathfrak{I} = \frac{1}{2} \frac{\mathfrak{I}''}{\left(\mathfrak{I}'\right)^2}$$

and integrating, we get

$$\int_0^z \mathfrak{I}(t)dt = \frac{1}{2}\left[-\frac{1}{\mathfrak{I}'(z)} + \frac{1}{\mathfrak{I}'(0)}\right] = \frac{1}{2}\left[-\frac{1}{\mathfrak{I}'(z)} + \frac{2}{\sqrt{\pi}}\right]$$

and multiplying by $\mathfrak{I}'(z)$ we obtain (2.5). □

PROPOSITION 2.3. *The derivatives of $\mathfrak{I}(z)$ satisfy the nonlinear recurrence*

$$d_{n+1} = \sqrt{\pi} \sum_{k=0}^{n-1} \binom{n}{k+1} d_k d_{n-k}, \quad n = 1, 2, \ldots \tag{2.6}$$

*with $d_0 = 0$ and $d_1 = \frac{\sqrt{\pi}}{2}$.*

PROOF. Using

$$\mathfrak{I}(z) = \sum_{n=0}^{\infty} d_n \frac{z^n}{n!}$$

and $d_1 = \frac{\sqrt{\pi}}{2}$ in (2.5), we have

$$\left[ \frac{\sqrt{\pi}}{2} + \sum_{n=1}^{\infty} d_{n+1} \frac{z^n}{n!} \right] \left[ \sum_{n=1}^{\infty} d_{n-1} \frac{z^n}{n!} - \frac{1}{\sqrt{\pi}} \right] = -\frac{1}{2}$$

or

$$\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} d_{n-1} \frac{z^n}{n!} + \sum_{n=2}^{\infty} \left[ \sum_{k=0}^{n-2} \binom{n}{k+1} d_k d_{n-k} \right] \frac{z^n}{n!} - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} d_{n+1} \frac{z^n}{n!} = 0.$$

Comparing powers of $z^n$, we get

$$\frac{\sqrt{\pi}}{2} d_{n-1} + \sum_{k=0}^{n-2} \binom{n}{k+1} d_k d_{n-k} - \frac{1}{\sqrt{\pi}} d_{n+1} = 0$$

or

$$\sum_{k=0}^{n-1} \binom{n}{k+1} d_k d_{n-k} - \frac{1}{\sqrt{\pi}} d_{n+1} = 0.$$

□

Although one could use (2.6) to compute the higher derivatives of $\operatorname{inverf}(z)$, the nonlinearity of the recurrence makes it hard to analyze the asymptotic behavior of $d_n$ as $n \to \infty$. Instead, we shall use an alternative technique that we developed in [**Dom03a**] and we called the method of "nested derivatives". The following theorem contains the main result presented in [**Dom03a**].

THEOREM 2.4. *Let*

$$H(x) = h^{-1}(x), \quad f(x) = \frac{1}{h'(x)}, \quad z_0 = h(x_0), \quad |f(x_0)| \in (0, \infty).$$

*Then,*

$$H(z) = x_0 + f(x_0) \sum_{n=1}^{\infty} \mathfrak{D}^{n-1}[f](x_0) \frac{(z - z_0)^n}{n!},$$

*where we define $\mathfrak{D}^n[f](x)$, the $n^{th}$ nested derivative of the function $f(x)$, by $\mathfrak{D}^0[f](x) = 1$ and*

$$\mathfrak{D}^{n+1}[f](x) = \frac{d}{dx} [f(x) \times \mathfrak{D}^n[f](x)], \quad n = 0, 1, \ldots. \tag{2.7}$$

The following proposition makes the computation of $\mathfrak{D}^{n-1}[f](x_0)$ easier in some cases.

Proposition 2.5. *Let*

$$\mathfrak{D}^n[f](x)=\sum_{k=0}^{\infty}A_k^n\frac{(x-x_0)^k}{k!},\qquad f(x)=\sum_{k=0}^{\infty}B_k\frac{(x-x_0)^k}{k!}. \tag{2.8}$$

*Then,*

$$A_k^{n+1}=(k+1)\sum_{j=0}^{k+1}A_{k+1-j}^nB_j. \tag{2.9}$$

Proof. From (2.8) we have

$$f(x)\mathfrak{D}^n[f](x)=\sum_{k=0}^{\infty}\alpha_k^n\frac{(x-x_0)^k}{k!}, \tag{2.10}$$

with

$$\alpha_k^n=\sum_{j=0}^{k}A_{k-j}^nB_j. \tag{2.11}$$

Using (2.8) and (2.10) in (2.7), we obtain

$$\sum_{k=0}^{\infty}A_k^{n+1}(x-x_0)^k=\frac{d}{dx}\sum_{k=0}^{\infty}\alpha_k^n(x-x_0)^k=\sum_{k=0}^{\infty}(k+1)\,\alpha_{k+1}^n(x-x_0)^k$$

and the result follows from (2.11). □

To obtain a linear relation between successive nested derivatives, we start by establishing the following lemma.

Lemma 2.6. *Let*

$$g_n(x)=\frac{\mathfrak{D}^n[f](x)}{f^n(x)}. \tag{2.12}$$

*Then,*

$$g_{n+1}(x)=g_n'(x)+(n+1)\frac{f'(x)}{f(x)}g_n(x),\quad n=0,1,\ldots. \tag{2.13}$$

Proof. Using (2.7) in (2.12), we have

$$g_{n+1}(x)=\frac{\mathfrak{D}^{n+1}[f](x)}{f^{n+1}(x)}=\frac{\frac{d}{dx}\left[f(x)\times\mathfrak{D}^n[f](x)\right]}{f^{n+1}(x)}$$
$$=\frac{\frac{d}{dx}\left[g_n(x)\,f^{n+1}(x)\right]}{f^{n+1}(x)}=\frac{g_n'(x)\,f^{n+1}(x)+g_n(x)\,(n+1)f^n(x)\,f'(x)}{f^{n+1}(x)}$$

and the result follows. □

Corollary 2.7. *Let*

$$H(x)=h^{-1}(x),\quad f(x)=\frac{1}{h'(x)},\quad z_0=h(x_0),\quad |f(x_0)|\in(0,\infty).$$

*Then,*

$$\frac{d^nH}{dz^n}(z_0)=[f(x_0)]^n\,g_{n-1}(x_0),\quad n=1,2,\ldots. \tag{2.14}$$

For the function $h(x) = \operatorname{erf}(z)$, we have

$$f(x) = \frac{1}{h'(x)} = \frac{\sqrt{\pi}}{2} \exp\left(x^2\right), \tag{2.15}$$

and setting $x_0 = 0$ we obtain $z_0 = \operatorname{erf}(0) = 0$. Using the Taylor series

$$\frac{\sqrt{\pi}}{2} \exp\left(x^2\right) = \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

in (2.9), we get

$$A_k^{n+1} = \frac{\sqrt{\pi}}{2} (k+1) \sum_{j=0}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \frac{A_{k+1-2j}^n}{j!},$$

with $A_k^n$ defined in (2.8). Using (2.15) in (2.13), we have

$$g_{n+1}(x) = g_n'(x) + 2(n+1) x g_n(x), \quad n = 0, 1, \ldots, \tag{2.16}$$

while (2.14) gives

$$d_n = \left(\frac{\sqrt{\pi}}{2}\right)^n g_{n-1}(0), \quad n = 1, 2, \ldots. \tag{2.17}$$

In the next section we shall find an asymptotic approximation for a family of polynomials closely related to $g_n(x)$.

## 3. The polynomials $P_n(x)$

We define the polynomials $P_n(x)$ by $P_0(x) = 1$ and

$$P_n(x) = g_n\left(\frac{x}{\sqrt{2}}\right) 2^{-\frac{n}{2}}. \tag{3.1}$$

$$P_{n+1}(x) = P_n'(x) + (n+1) x P_n(x), \tag{3.2}$$

The first few $P_n(x)$ are

$$P_1(x) = x, \quad P_2(x) = 1 + 2x^2, \quad P_3(x) = 7x + 6x^3, \ \ldots \ .$$

The following propositions describe some properties of $P_n(x)$.

Proposition 3.1. *Let*

$$P_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} C_k^n x^{n-2k}, \tag{3.3}$$

*where* $\lfloor \cdot \rfloor$ *denotes the integer part function. Then,*

$$C_0^n = n! \tag{3.4}$$

*and*

$$C_k^n = n! \sum_{j_k=0}^{n-1} \sum_{j_{k-1}=0}^{j_k-1} \cdots \sum_{j_1=0}^{j_2-1} \prod_{i=1}^{k} \frac{j_i - 2i + 2}{j_i + 1}, \quad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor. \tag{3.5}$$

Proof. Using (3.3) in (3.2) we have

$$\sum_{0\le 2k\le n+1} C_k^{n+1} x^{n+1-2k} = \sum_{0\le 2k\le n} C_k^n (n-2k)\, x^{n-2k-1} + \sum_{0\le 2k\le n} (n+1)\, C_k^n x^{n+1-2k}$$
$$= \sum_{2\le 2k\le n+2} C_{k-1}^n (n-2k+2)\, x^{n+1-2k} + \sum_{0\le 2k\le n} (n+1)\, C_k^n x^{n+1-2k}.$$

Comparing coefficients in the equation above, we get

$$C_0^{n+1} = C_0^n, \tag{3.6}$$

$$C_k^{n+1} = (n-2k+2)\, C_{k-1}^n + (n+1)\, C_k^n, \quad k = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \tag{3.7}$$

and for $n = 2m-1$,

$$C_m^{2m} = C_{m-1}^{2m-1}, \quad m = 1, 2, \ldots.$$

From (3.6) we immediately conclude that $C_0^n = n!$, while (3.7) gives

$$C_k^n = n! \sum_{j=0}^{n-1} \frac{j-2k+2}{(j+1)!} C_{k-1}^j, \quad n, k \ge 1. \tag{3.8}$$

Setting $k = 1$ in (3.8) and using (3.4), we have

$$C_1^n = n! \sum_{j=0}^{n-1} \frac{j}{(j+1)!} C_0^j = n! \sum_{j=0}^{n-1} \frac{j}{j+1}. \tag{3.9}$$

Similarly, setting $k = 2$ in (3.8) and using (3.9), we get

$$C_2^n = n! \sum_{j=0}^{n-1} \frac{j-2}{(j+1)!} \left[ j! \sum_{i=0}^{j-1} \frac{i}{i+1} \right] = n! \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} \frac{j-2}{j+1} \frac{i}{i+1}$$

and continuing this way we obtain (3.5). □

Proposition 3.2. *The zeros of the polynomials $P_n(x)$ are purely imaginary for $n \ge 1$.*

Proof. For $n = 1$ the result is obviously true. Assuming that it is true for $n$ and that $P_n(x)$ is written in the form

$$P_n(x) = n! \prod_{k=1}^{n} (z - z_k), \quad \mathrm{Re}(z_k) = 0, \quad 1 \le k \le n, \tag{3.10}$$

we have two possibilities for $z^*$, with $P_{n+1}(z^*) = 0$:

(1) $z^* = z_k$, for some $1 \le k \le n$.
In this case, $\mathrm{Re}(z^*) = 0$ and the proposition is proved.

(2) $z^* \ne z_k$, for all $1 \le k \le n$.
From (3.2) and (3.10) we get

$$\frac{P_{n+1}(x)}{P_n(x)} = \frac{d}{dx} \ln\left[P_n(x)\right] + (n+1)x = \sum_{k=1}^{n} \frac{1}{z - z_k} + (n+1)x.$$

Evaluating at $z = z^*$, we obtain

$$0 = \sum_{k=1}^{n} \frac{1}{z^* - z_k} + (n+1)z^*$$

and taking Re(•), we have

$$0 = \mathrm{Re}\left[\sum_{k=1}^{n} \frac{1}{z^* - z_k} + (n+1)z^*\right]$$

$$= \sum_{k=1}^{n} \frac{\mathrm{Re}\,(z^* - z_k)}{\left|z^* - z_k\right|^2} + (n+1)\,\mathrm{Re}(z^*) = \mathrm{Re}(z^*)\left[\sum_{k=1}^{n} \frac{1}{\left|z^* - z_k\right|^2} + n + 1\right]$$

which implies that $\mathrm{Re}(z^*) = 0$.

□

**3.1. Asymptotic analysis of $P_n(x)$.** We first consider solutions to (3.2) of the form

$$P_n(x) = n! A^{(n+1)}(x), \tag{3.11}$$

with $x > 0$. Replacing (3.11) in (3.2) and simplifying the resulting expression, we obtain

$$A^2(x) = A'(x) + xA(x),$$

with solution

$$A(x) = \exp\left(-\frac{x^2}{2}\right)\left[C - \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]^{-1}, \tag{3.12}$$

for some constant $C$. Note that (3.11) is not an exact solution of (3.2), since it does not satisfy the initial condition $P_0(x) = 1$. To determine $C$ in(3.12), we observe from (3.4) that

$$P_n(x) \sim n! x^n, \quad x \to \infty. \tag{3.13}$$

As $x \to \infty$, we get from (3.12)

$$\ln\left[A(x)\right] \sim -\frac{x^2}{2} - \ln\left(C - \sqrt{\frac{\pi}{2}}\right) + \frac{\exp\left(-\frac{x^2}{2}\right)}{\left(C - \sqrt{\frac{\pi}{2}}\right)x}, \quad x \to \infty,$$

which is inconsistent with (3.13) unless $C = \sqrt{\frac{\pi}{2}}$. In this case, we have

$$A(x) \sim x + \frac{1}{x}, \quad x \to \infty, \tag{3.14}$$

matching (3.13). Thus,

$$A(x) = \sqrt{\frac{2}{\pi}}\exp\left(-\frac{x^2}{2}\right)\left[1 - \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]^{-1}. \tag{3.15}$$

Since (3.11) and (3.14) give

$$P_n(x) \sim n! x^{n+1}, \quad x \to \infty,$$

instead of (3.13), we need to consider

$$P_n(x) = n! A^{(n+1)}(x) B(x, n). \tag{3.16}$$

Replacing (3.16) in (3.2) and simplifying, we get

$$B(x, n+1) = B(x, n) + \frac{1}{A(x)(n+1)}\frac{\partial B}{\partial x}(x, n).$$

Using the approximation

$$B(x,n+1) = B(x,n) + \frac{\partial B}{\partial n}(x,n) + \frac{1}{2}\frac{\partial^2 B}{\partial n^2}(x,n) + \cdots ,$$

we obtain

$$\frac{\partial B}{\partial n} = \frac{1}{A(x)(n+1)}\frac{\partial B}{\partial x},$$

whose solution is

$$B(x,n) = F\left[\frac{n+1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right], \tag{3.17}$$

for some function $F(u)$. Matching (3.16) with (3.13) requires

$$B(x,n) \sim \frac{1}{x}, \quad x \to \infty. \tag{3.18}$$

Since in the limit as $x \to \infty$, with $n$ fixed we have

$$\ln\left[\frac{n+1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right] \sim \frac{x^2}{2},$$

(3.17)-(3.18) imply

$$F(u) = \frac{1}{\sqrt{2\ln(u)}}.$$

Therefore, for $x > 0$,

$$P_n(x) \sim n!\Phi(x,n), \quad n \to \infty, \tag{3.19}$$

with

$$\Phi(x,n) = \left[\sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{x^2}{2}\right)}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right]^{n+1}\left[2\ln\left(\frac{n+1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right)\right]^{-\frac{1}{2}}.$$

From (3.3) we know that the polynomials $P_n(x)$ satisfy the reflection formula

$$P_n(-x) = (-1)^n P_n(x). \tag{3.20}$$

Using (3.20), we can extend (3.19) to the whole real line and write

$$P_n(x) \sim n!\left[\Phi(x,n) + (-1)^n \Phi(-x,n)\right], \quad n \to \infty. \tag{3.21}$$

In Figure 1 we compare the values of $P_{10}(x)$ with the asymptotic approximation (3.21).

We see that the approximation is very good, even for small values of $n$. We summarize our results of this section in the following theorem.

THEOREM 3.3. *Let the polynomials $P_n(x)$ be defined by*

$$P_{n+1}(x) = P_n'(x) + (n+1)\,xP_n(x),$$

*with $P_0(x) = 1$. Then, we have*

$$P_n(x) \sim P_n(x) \sim n!\left[\Phi(x,n) + (-1)^n \Phi(-x,n)\right], \quad n \to \infty, \tag{3.22}$$

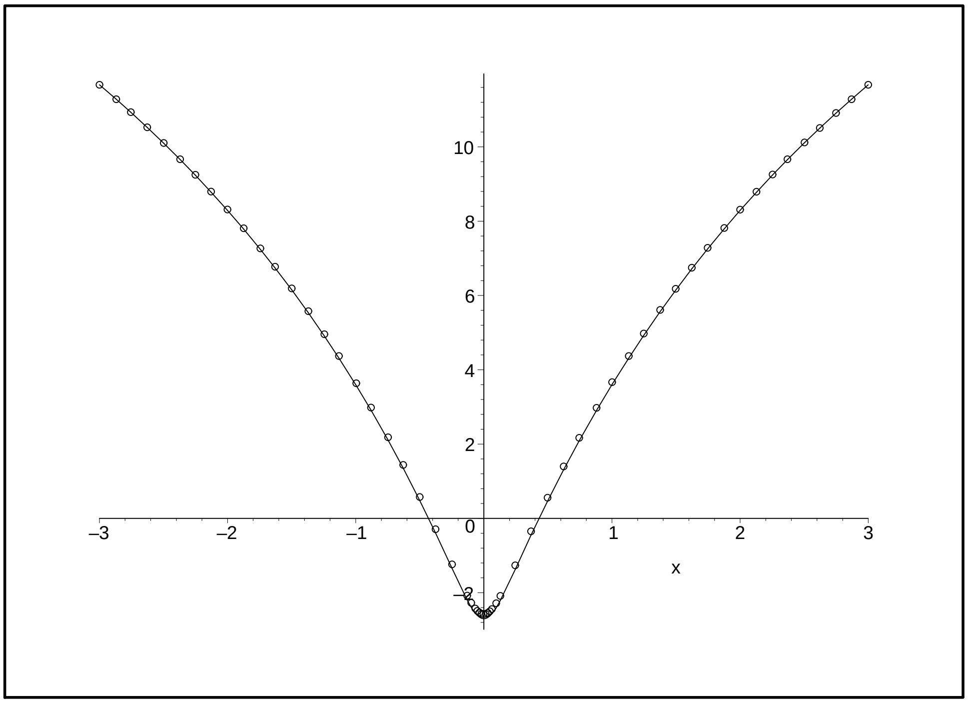


FIGURE 1. A sketch of the exact (solid curve) and asymptotic (ooo) values of $\ln\left[\frac{P_{10}(x)}{10!}\right]$.

*where*

$$\Phi(x,n)=\left[\sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{x^2}{2}\right)}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right]^{n+1}\left[2\ln\left(\frac{n+1}{1-\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)}\right)\right]^{-\frac{1}{2}}. \tag{3.23}$$

## 4. Higher derivatives of $\operatorname{inverf}(z)$

From (2.17) and (3.1), it follows that

$$d_n=\frac{1}{\sqrt{2}}\left(\sqrt{\frac{\pi}{2}}\right)^n P_{n-1}(0), \quad n=1,2,\ldots, \tag{4.1}$$

where $d_n$ was defined in (2.1). Using Theorem 3.3 in (4.1), we have

$$d_n\sim\frac{1}{\sqrt{2}}\left(\sqrt{\frac{\pi}{2}}\right)^n\Phi(0,n-1)\left[1+(-1)^{n-1}\right],$$

as $n\to\infty$. Using (3.23), we obtain

$$\frac{d_n}{n!}\sim\frac{1}{2n\sqrt{\ln(n)}}\left[1+(-1)^{n-1}\right], \quad n\to\infty. \tag{4.2}$$

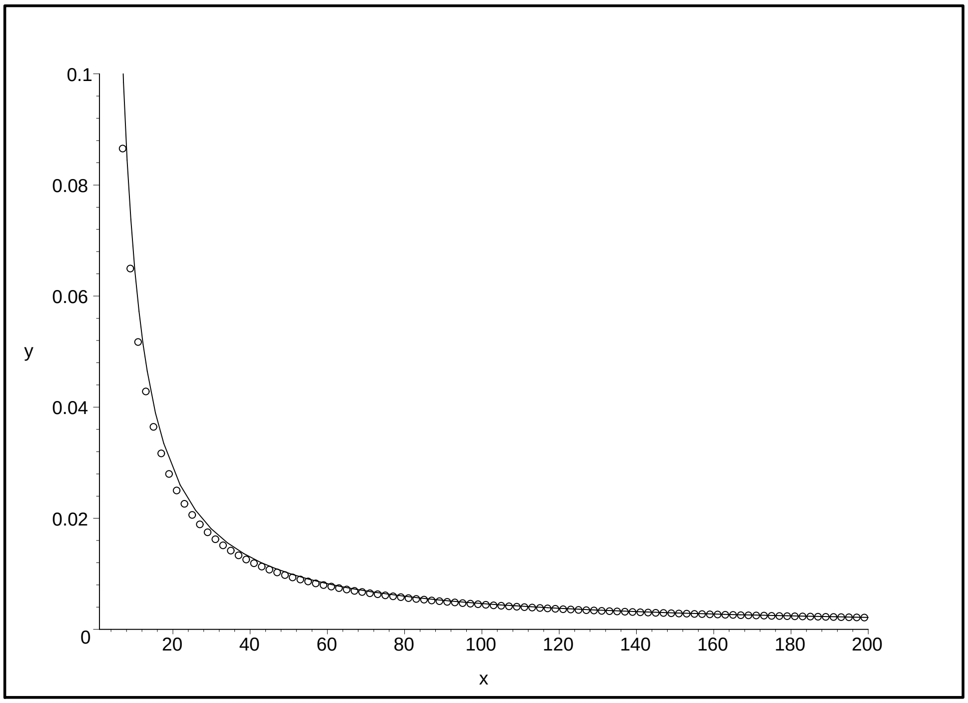


FIGURE 2. A sketch of the exact (ooo) and asymptotic (solid curve) values of $\frac{d_{2k+1}}{(2k+1)!}$.

Setting $n = 2N + 1$ in (4.2), we have

$$\frac{d_{2N+1}}{(2N+1)!} \sim \frac{1}{(2N+1)\sqrt{\ln(2N+1)}}, \quad N \to \infty. \tag{4.3}$$

**4.1. Numerical results.** In this section we demonstrate the accuracy of the approximation (4.2) and construct a high order Taylor series for $\operatorname{inverf}(x)$. In Figure 2 we compare the logarithm of the exact values of $\frac{d^{2n+1}}{dz^{2n+1}} \operatorname{inverf}(x)\Big|_{x=0}$ and our asymptotic formula (4.2). We see that there is a very good agreement, even for moderate values of $n$.

Using (2.6), we compute the exact values

$$d_1 = \frac{1}{2}\pi^{\frac{1}{2}}, \quad d_3 = \frac{1}{4}\pi^{\frac{3}{2}}, \quad d_5 = \frac{7}{8}\pi^{\frac{5}{2}}, \quad d_7 = \frac{127}{16}\pi^{\frac{7}{2}}, \quad d_9 = \frac{4369}{32}\pi^{\frac{9}{2}}$$

and form the polynomial Taylor approximation

$$T_9(x) = \sum_{k=0}^{4} d_{2k+1} \frac{x^{2k+1}}{(2k+1)!}.$$

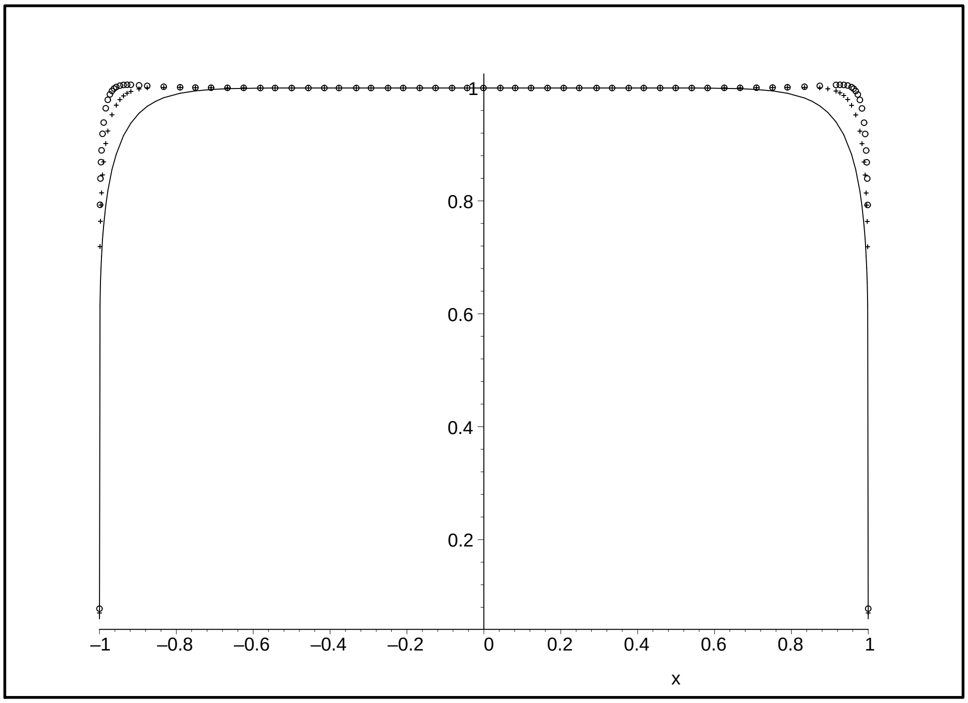


FIGURE 3. A sketch of $\frac{T_9(x)}{\mathrm{inverf}(x)}$ (solid curve), $\frac{T_9(x)+R_{10}(x)}{\mathrm{inverf}(x)}$ (+++) and $\frac{T_9(x)+R_{20}(x)}{\mathrm{inverf}(x)}$ (ooo).

In Figure 3 we graph $\frac{T_9(x)}{\mathrm{inverf}(x)}$ and $\frac{T_9(x)+R_N(x)}{\mathrm{inverf}(x)}$, for $N = 10, 20$, where

$$R_N(x) = \sum_{k=5}^{N} \frac{x^{2k+1}}{(2N+1)\sqrt{\ln(2N+1)}}, \quad N = 5, 6, \ldots. \tag{4.4}$$

The functions are virtually identical in most of the interval $(-1, 1)$ except for values close to $x = \pm 1$. We show the differences in detail in Figure 4. Clearly, the additional terms in $R_{20}(x)$ give a far better approximation for $x \simeq 1$.

In the table below we compute the exact value of and optimal asymptotic approximation to inverf $(x)$ for some $x$:

| $x$ | inverf $(x)$ | $T_9(x) + R_N(x)$ | $N$ |
|---|---|---|---|
| 0.7 | .732869 | .732751 | 6 |
| 0.8 | .906194 | .905545 | 7 |
| 0.9 | 1.16309 | 1.16274 | 11 |
| 0.99 | 1.82139 | 1.82121 | 57 |
| 0.999 | 2.32675 | 2.32676 | 423 |
| 0.9999 | 2.75106 | 2.75105 | 3685 |

.

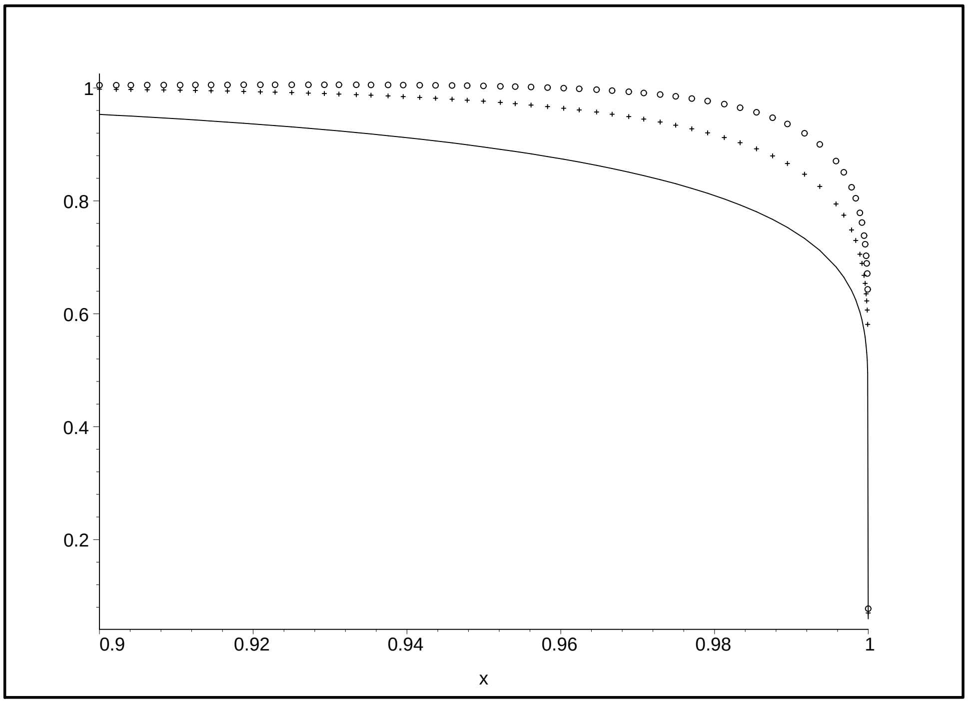


FIGURE 4. A sketch of $\frac{T_9(x)}{\text{inverf}(x)}$ (solid curve), $\frac{T_9(x)+R_{10}(x)}{\text{inverf}(x)}$ (+++) and $\frac{T_9(x)+R_{20}(x)}{\text{inverf}(x)}$ (ooo).

Clearly, (4.4) is still valid for $x \to 1$, but at the cost of having to compute many terms in the sum. In this region it is better to use the formula [**Dom03b**]

$$\text{inverf}\,(x) \sim \sqrt{\frac{1}{2}\,\text{LW}\left[\frac{2}{\pi\,(x-1)^2}\right]}, \quad x \to 1^-,$$

where LW$(\cdot)$ denotes the Lambert-W function [**CGH$^+$96**], which satisfies

$$\text{LW}(x)\exp\left[\text{LW}(x)\right] = x.$$

## References

[Ber89] M. V. Berry, *Uniform asymptotic smoothing of Stokes's discontinuities*, Proc. Roy. Soc. London Ser. A **422** (1989), no. 1862, 7–21. MR MR990851 (90h:34084)

[BKSH99] D. J. Brabander, N. Keon, R. H. R. Stanley, and H. F. Hemond, *Intra-ring variability of Cr, As, Cd, and Pb in red oak revealed by secondary ion mass spectrometry: Implications for environmental biomonitoring*, Proc. Natl. Acad. Sci. USA **96** (1999), no. 25, 14635–14640. MR MR2142222

[Car63] L. Carlitz, *The inverse of the error function*, Pacific J. Math. **13** (1963), 459–470. MR MR0153878 (27 #3839)

[CC96] Ovidiu Costin and Rodica Costin, *Rigorous WKB for finite-order linear recurrence relations with smooth coefficients*, SIAM J. Math. Anal. **27** (1996), no. 1, 110–134. MR MR1373150 (97a:39005)

[CGH+96] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *On the Lambert W function*, Adv. Comput. Math. **5** (1996), no. 4, 329–359. MR MR1414285 (98j:33015)

[Dom03a] Diego Dominici, *Nested derivatives: a simple method for computing series expansions of inverse functions*, Int. J. Math. Math. Sci. (2003), no. 58, 3699–3715. MR MR2031140 (2005f:41079)

[Dom03b] Diego E. Dominici, *The inverse of the cumulative standard normal probability function*, Integral Transforms Spec. Funct. **14** (2003), no. 4, 281–292. MR MR1986919 (2004d:60027)

[Fet74] Henry E. Fettis, *A stable algorithm for computing the inverse error function in the "tail-end" region*, Math. Comp. **28** (1974), 585–587. MR MR0341812 (49 #6558)

[Her88] Gerhard Herden, *The role of error-functions in order to obtain relatively optimal classification*, Classification and related methods of data analysis (Aachen, 1987), North-Holland, Amsterdam, 1988, pp. 105–111. MR MR999553

[Jae46] J. C. Jaeger, *Some applications of the repeated integrals of the error function*, Quart. Appl. Math. **4** (1946), 100–103. MR MR0016873 (8,81f)

[Loz96] Daniel W. Lozier, *Software needs in special functions*, Proceedings of the Sixth International Congress on Computational and Applied Mathematics (Leuven, 1994), vol. 66, 1996, pp. 345–358. MR MR1393742

[LW95] James E. Lidsey and I. Waga, *Andante regime of scalar field dynamics in the chaotic inflationary universe*, Phys. Rev. D **51** (1995), no. 2, 444–449.

[Olv97] Frank W. J. Olver, *Asymptotics and special functions*, AKP Classics, A K Peters Ltd., Wellesley, MA, 1997, Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)]. MR MR1429619 (97i:41001)

[Phi55] J. R. Philip, *Numerical solution of equations of the diffusion type with diffusivity concentration-dependent*, Trans. Faraday Soc. **51** (1955), 885–892. MR MR0071876 (17,196f)

[Sha73] L. F. Shampine, *Exact solutions for concentration dependent diffusion and the inverse complementary error function*, J. Franklin Inst. **295** (1973), 239–247. MR MR0281322 (43 #7040)

[SJEMFAL+03] Ral San Jos Estpar, Marcos Martn-Fernndez, Carlos Alberola-Lpez, James Ellsmere, Ron Kikinis, and Carl-Fredrik Westin, *Freehand ultrasound reconstruction based on ROI prior modeling and normalized convolution*, Sixth International Conference on Medical Image Computing and Computer-Assisted Intervention (MICCAI'03), Lecture Notes in Computer Science 2879 (Montreal, Canada) (R. E. Ellis and T. M. Peters, eds.), Springer Verlag, November 15–18 2003, pp. 382–390.

[TM96] Kudo T. and Hibino M., *Consideration on the potential-composition relationships observed with amorphous intercalation systems such as lixwo3*, Solid State Ionics **84** (1996), no. 1, 65–72. MR MR2166352 (2006c:86006)

[Wal50] Henry Wallman, *Transient response and the central limit theorem of probability*, Proc. Symposia Appl. Math. **2** (1950), 91. MR MR0034250 (11,564e)

[WM99] Claus O. Wilke and Thomas Martinetz, *Lifetimes of agents under external stress*, Phys. Rev. E **59** (1999), no. 3, R2512–R2515.


Department of Mathematics, State University of New York at New Paltz, 75 S. Manheim Blvd. Suite 9, New Paltz, NY 12561-2443, USA, Phone: (845) 257-2607, Fax: (845) 257-3571

*E-mail address*: `dominicd@newpaltz.edu`